J419. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^4+b+c^4}+\frac{1}{b^4+c+a^4}+\frac{1}{c^4+a+b^4} \le \frac{3}{a+b+c}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

In homogeneous form the original inequality becomes

$$\sum_{cyc} \frac{abc}{a^4+ab^2c+c^4} \le \frac{3}{a+b+c}.$$

Since

$$\frac{abc}{a^4+ab^2c+c^4} = \frac{b}{\dfrac{a^3}{c}+b^2+\dfrac{c^3}{a}} \text{ and } \frac{a^3}{c}+\frac{c^3}{a} \ge a^2+b^2 \iff a^4+c^4 \ge a^3c+ac^3 \iff$$

$$\left(a^2+ac+c^2\right)(a-c)^2 \ge 0$$

then

$$\frac{b}{\dfrac{a^3}{c}+b^2+\dfrac{c^3}{a}} \le \frac{b}{a^2+b^2+c^2}$$

and, therefore,

$$\sum_{cyc} \frac{abc}{a^4+ab^2c+c^4} \le \sum_{cyc} \frac{b}{a^2+b^2+c^2} = \frac{a+b+c}{a^2+b^2+c^2}.$$

And also we have

$$\frac{a+b+c}{a^2+b^2+c^2} \le \frac{3}{a+b+c} \iff (a+b+c)^2 \le 3\left(a^2+b^2+c^2\right) \iff$$

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Polyahedra, Polk State College, FL, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Konstantinos Metaxas, Athens, Greece; Kevin Soto Palacios, Huarmey-Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Abhay Chandra, Indian Institute of Technology, New Delhi, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România.